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Gravitating Dyons and Dyonic Black Holes in SU(2) and SU(5) Theories

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Abstract The study of gravitating dyons and dyonic black holes in SU(2) and SU(5) theories has been undertaken and it has been shown that gravitating fundamental dyonic solutions and dyonic black holes are stable in both the cases.

Keywords Dyons · Dyonic black holes · Non-Abelian gauge theories · R-N black holes

1 Introduction

The rapid progress in the theory of gravitating solutions and hairy black holes started after Bartnik and Mc Kinnon discovered [1] the particle like solutions (BK Solutions) in four dimensional Einstein-Yang-Mills (EYM) theory of the gauge group SU(2). This came as a big surprise since it is well known that when taken apart, neither pure gravity nor YM theory admits particle like solutions. Soon after BK discovery it was realized [2, 3] that the EYM model contains also the non-Abelian black holes apart from solutions and consequently some basic concepts of black hole physics based on the uniqueness and no-hair therem [4–6] were revised under the violation of no-hair conjecture which is typical for gravitating non-Abelian gauge theories. The EYM black-holes uniquely distinguish themselves from solutions of other gravitating non-Abelian models by their peculiar oscillatory behaviour in the interior region, which is reminiscent of cosmological models [5, 6].

The external structure of EYM black hole solutions does not change considerably [7] after adding some additional matter like Higgs fields but the solutions in the interior region change completely [8, 9] and exhibit a regular power law behaviour near the singularity. Static and spherically symmetrical solutions of EYMH system have been intensively studied [10–17] to understand the nature of black holes, specially in the context of no-hair conjecture and also to understand the properties of point like solutions like regular monopoles

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under gravity. The dynamics of gravitating monopoles has been worked out [10, 11, 18] analytically as well as numerically in SU(2) gauge theory revealing a number of interesting phenomena for gravitating monopoles and black holes when the strength of gravitational force is either very weak or becomes comparable to that of YM interaction. It has also been demonstrated that besides the gravitating monopole solutions having the flat limit, there also exists a discrete family of radially excited monopole solutions having no counter part in flat space. Another new feature without analogue in flat space is the existence of non-Abelian magnetically charged black holes.

Keeping in view that in SU(2) case the gravitating solutions representing static monopole—anti monopole pairs are at most axially symmetric [19] consistent with the flat limit [20], recently SU(5) gravitating solutions (including monopoles and black holes) have been derived as embeddings of SU(2) ones [21] as well as the solutions [22] of full second order EMH equations coupled with gravity, which are neither the embeddings of SU(2) ones nor the solutions of first order Bogomolnyi equations.

Static spherically symmetric gravitating dyonic solutions and dyonic black holes distinct from embedded Reissner-Norsdrom (R-N) black holes with unit magnetic charge and arbitrary electric charge, have been studied [23] in EYMH theory in SU(2) case and it has been shown that the gravitating dyon solutions share many features with gravitating monopole solutions and also that the classical dyonic solution in flat space is unstable.

In the present papers we have undertaken the study of gravitating dyons and dyonic black holes in SU(2) and SU(5) theories by taking the generalized charge of a dyon as a complex quantity with its real and imaginary parts as electric and magnetic constituents. Modifying the results of Brihaye et al. [23] for SU(2) case it has been demonstrated that the gravitating fundamental dyonic solutions may also be stable since the mass involved can not always be lowered continuously by lowering the electric charge. Following the approach of Yu [24], spherically symmetric dyonic solutions of the full second order SU(5) EYMH equations have been constructed, asymptotically flat metric of dyons has been obtained and the field equations have been complemented with suitable boundary conditions for getting the solutions with regular origin or black holes. It has been shown that in both these cases the corresponding boundary points are singular points of the field equations. It has also been shown that SU(5) gravitating dyon also bifurcates into an extremal Reisser-Nordstrom black hole corresponding to the solution of Abelian-Einstein-Maxwell equations, which can be embedded into the non-Abelian one with the mass which can not always be lowered continuously by lowering the electric charge in SU(5) theory also.

2 Gravitating Dyons in SU(2) Theory

A general non-Abelian gauge theory of dyons (Yang-Mills-Higgs Theory) consists of usual four-space (external) and n-dimensional internal group space, where the field associated with dyons has n-fold internal multiplicity and the multiplets of gauge field transform as a basis of adjoint representation of n-dimensional non-Abelian gauge symmetry group. Choosing the internal gauge group as SU(2), the generalized dyonic field tensor may be constructed as [25]

$$\vec{G}_{\mu\upsilon} = G^a_{\mu\upsilon} T_a, \tag{2.1}$$

with the generalized four-potential defined as

$$\vec{V}_\mu = V^a_\mu T_a, \tag{2.2}$$

where the vector sign is denoted in the internal group space; $\mu, \upsilon = 0, 1, 2, 3$ represent the degrees of freedom in the external space and matrices T_a, with $a = 1, 2, 3$, are the infinitesimal generators of group SU(2). $\vec{G}_{\mu\upsilon}$ and $\vec{V}_\mu$ are connected as

$$\vec{G}_{\mu\upsilon} = \partial_\mu \vec{V}_\upsilon - \partial_\upsilon \vec{V}_\mu + |q|[\vec{V}_\mu, \vec{V}_\upsilon]$$

which may also be written as

$$G^a_{\mu\upsilon} = \partial_\mu V^a_\upsilon - \partial_\upsilon V^a_\mu + |q|\varepsilon^{abc} V_{\mu b} V_{\upsilon c} \tag{2.3}$$

$$\text{where } q = e - ig, \tag{2.4}$$

is the dyonic generalized charge with e and g as electric and magnetic constituents. In order to study the effect of gravity on dyons in SU(2) theory, let us start with the following Einstein-Yang-Mills-Higgs (FYMH) action [23]

$$S = \int \left[\frac{R}{16\pi G} - \frac{1}{4} G^a_{\mu\upsilon} G^{\mu\upsilon}_a + (D_\mu \phi^a)(D^\mu \phi_a) - V(\phi) \right] \sqrt{-g'} d^4x \tag{2.5}$$

where SU(2) potential is

$$V(\phi) = (\lambda/4)(\phi^a \phi_a)^2 - (\lambda/2) \upsilon^2 (\phi_a \phi^a) - V_{\min} \tag{2.6}$$

with $V_{\min} = (\lambda/4)\upsilon^4$; λ is Higgs self coupling constant, $\upsilon = \langle \Phi \rangle_0$ is the vacuum expectation value of Higgs field Φ and g' is the determinant of the spherically symmetric Schwarzschild-like metric

$$ds^2 = -A^2(r)\mu(r)dt^2 + \underline{dr^2} + r^2(d\theta^2 + \sin^2\theta d\Phi^2) \tag{2.7}$$

$$\text{with } \mu(r) = 1 - 2\frac{m(r)}{r}. \tag{2.8}$$

The variations of the action (2.5) with $g_{\mu\upsilon}$, gauge field V^a_μ and Higgs field Φ^a lead to Einstein equations and matter field equations. In this action the YM fields are purely repulsive and Higg's fields are purely attractive and these repulsion and attraction can compensate each other leading to the existence of stable dyonic solutions.

Let us impose the following ansatz [26] on the gauge and Higgs fields;

$$\begin{aligned} V_{ia} &= \varepsilon_{aij}(r^j/r^2)[K(r) - I]/|q|, \\ V_{oa} &= r_a J(r)/(r|q|) \quad \text{and} \\ \Phi^a &= r^a H(r)/(r|q|) \end{aligned} \tag{2.9}$$

where the functions $H(r)$, $J(r)$ and $K(r)$ satisfy the following radial differential equations;

$$\begin{aligned} r^2 H''(r) &= 2HK^2, \\ r^2 J''(r) &= 2JK^2 \quad \text{and} \\ r^2 K''(r) &= K(K^2 - 1) + K(H^2 - J^2) \end{aligned} \tag{2.10}$$

where $'$ denotes the derivative with respect to r.

In Prasad Sommerfield limit [27], we have

$$V(\Phi) = 0$$

but

$$\langle\Phi\rangle = v \neq 0.$$

In this limit, the dyons have lowest possible energy for given electric and magnetic charges e and g respectively and then we get the following expression for dyonic mass

$$M_D = v(e^2 + g^2)^{1/2} = v|q| \tag{2.11}$$

where the electric and magnetic fields associated with dyons obey the first order equations

$$\begin{aligned} E_i^a &= G_{oi}^a = (D_i\Phi)^a \sin\gamma, \\ B_i^a &= 1/2\varepsilon_{ijk}G^{jka} = (D_j\Phi)^a \cos\gamma \qquad \text{and} \\ (D_0\Phi)^a &= 0 \end{aligned} \tag{2.12}$$

with

$$\gamma = \tan^{-1} e/g. \tag{2.13}$$

Using Gauss's law and these expressions for fields the electric and magnetic charges of a dyon may be written as

$$\begin{aligned} e &= (I/v)\int \partial_i(\phi^a G_{oi}^a)d^3x \qquad \text{and} \\ g &= 1/2v\int \varepsilon_{ijk}\partial_i(\Phi^a G_{jk}^a)d^3x. \end{aligned} \tag{2.14}$$

With these electric and magnetic charges on a dyon, the Dirac's quantization is modified to chirality quantization given by [28]

$$\mu_{ij} = e_i g_j - e_j g_i = n/2 \tag{2.14a}$$

where e_i and g_i are electric and magnetic charges on ith dyon, μ_{ij} is the magnetic coupling parameter [29] between ith and jth dyon and n is integer (rather even integer as required [30] by chiral invariance and locality of dyonic fields).

For the case of pure monopole, $\gamma = 0$ and $V^{oa} = 0$ and then from (2.12) we have

$$B_i = D_i\Phi \tag{2.15}$$

which is Bogomolnyi equation [31]. It does not allow static dyonic solutions but in this case the dyons emerge as time dependent solutions [25].

In spherical coordinate systems, used in (2.7) for metric, the ansatz (2.9) may be written as

$$\begin{aligned} &\vec{V}_0 = \hat{e}_r J(r)/|q|, \qquad \vec{V}_r = 0, \\ &\vec{V}_\theta = -\hat{e}_\Phi[1-K(r)]/|q|, \qquad \vec{V}_\Phi = \hat{e}_\theta \frac{[1-K(r)]}{|q|}\sin\theta \quad \text{and} \\ &\Phi = \hat{e}_r H(r)/|q| \end{aligned} \tag{2.16}$$

where $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\Phi$ are the unit vectors along spherical coordinates.

Let us induce two dimensionless parameters α and β through the mass ratios

$$\alpha = \frac{M_D}{|q|M_{pl}} \quad \text{and} \quad \beta = \frac{M_H}{M_D} \tag{2.17}$$

where $M_D = |q|v$ is the dyonic mass, $M_{pl} = 1/\sqrt{G}$ is Planck mass and $M_H = \sqrt{\lambda}v$. In other words

$$\alpha = v\sqrt{G} \quad \text{and} \quad \beta = \sqrt{\lambda}/|q|. \tag{2.17a}$$

Using these coupling constants and (2.7), (2.16) and (2.10), the following field equations follow from the action given by (2.5);

$$\mu'(r) = (1-\mu)/r - 2[\alpha^2/|q|^2v^2r][r^2J'^2 + J^2K^2/A^2\mu + \mu K'^2 + (1/2)\mu r^2H'^2 \\ + (K^2-1)^2/2r^2 + H^2K^2 + (\beta^2r^2/4)(H^2-|q|^2v^2)^2], \tag{2.18}$$

$$A' = \alpha^2 r/(v^2|q|^2)[2J^2K^2/(A^2\mu^2r) + 2K'^2/r^2 + H'^2]A, \tag{2.18a}$$

$$(A\mu K')' = AK[(K^2-1)/r^2 + H^2 - J^2/(A^2\mu)], \tag{2.18b}$$

$$(r^2J'/A)' = 2J/(|q|v)K^2/(A\mu), \tag{2.18c}$$

and

$$(r^2A_\mu H')' = AH/(v|q|)[2K^2 + \beta^2r^2(H^2 - v^2|q|^2)]. \tag{2.18d}$$

The following special solutions of these equations are of current physical interest;

(A) Embedded R-N Solutions

For the values of v (vacuum expectation values of Higgs field) more than critical value for the fixed value of the parameter β, the only static solution of (2.18) is the Reissner-Nordstrom black hole (R-N black hole). For mass μ_∞, unit magnetic charge and arbitrary electric charge e such embedded R-N solutions emerge with

$$\begin{aligned} &m(r) = m_\infty - \alpha^2/(2v^2r) = m_\infty - G/2r, \\ &A(r) = 1, \qquad K(r) = 0, \qquad J(r) = J_\infty - e/r \quad \text{and} \\ &H(r) = v[1+e^2)^{1/2}. \end{aligned} \tag{2.19}$$

The horizon of the corresponding external R-N solution is

$$r_H = m_\infty = \alpha/v = \sqrt{G}. \tag{2.20}$$

Since the Higgs field can be globally transformed into a fixed SU(2) direction, these relations correspond to an Abelian configuration.

(B) Dyonic Solutions

It is well known that SU(2) gauge theory coupled to a Higgs field in the adjoint representation admits in flat space the Julia Zee dyons [26]. For such asymmetrically flat solutions both the metric functions $A(r)$ and $m(r)$ of (2.7) and (2.8) approach constant values at infinity. Let us choose

$$\begin{aligned} &A(\infty) = 1 \quad \text{and} \\ &m(\infty) = m_\infty \end{aligned} \tag{2.21}$$

which represents the mass of the solutions. The asymptotic limits of the matter functions may be taken as

$$K(\infty)=0, \qquad J(\infty)=J_\infty \quad \text{and} \quad H(\infty)=v|q|. \tag{2.22}$$

For the regularity of solutions at the origin we have

$$m(0)=0 \quad \text{and} \tag{2.23}$$

$$K(0)=1, \qquad J(0)=0, \qquad H(0)=0. \tag{2.24}$$

With these conditions, the dyonic solutions in flat space, in the limit $\beta=0$, are very well known analytically [27]. The corresponding gravitating dyon solutions (in the presence of gravity) exist for $\alpha<\alpha_{\max}$ where $\alpha_{\max}$ is the maximal value of α obtained by varying G upto maximal value while keeping v fixed. Keeping G fixed at this maximal value and varying v upto its critical value, α reaches its critical value α_c for which the fundamental dyonic solution reaches to limiting form and bifurcates to the extremal R-N solutions, with unit magnetic charge and arbitrary electric charge e described by (2.19). For these fundamental dyonic solutions, we have

$$\alpha_c<\alpha<\alpha_{\max}. \tag{2.25}$$

In this limit there are two regular solutions while for $\alpha>\alpha_{\max}$ there is no regular solution and at $\alpha=0$, the flat space solutions are obtained. α may be made vanishing in two ways, either for $G\to 0$ with fixed v or for $v\to 0$ with G fixed. In first case we get flat space Julia-Zee dyons and in the second case one gets EYM theory. Regular dyons exist only for $\alpha<\alpha_{\max}$ where the precise value of $\alpha_{\max}$ depends on β given by (2.17a)

For $\alpha\to\alpha_c$ the fundamental dyonic solutions approach limiting functions and bifurcate with extremal R-N solutions of unit magnetic charge and arbitrary electric charge e i.e. $|q|=(1+e^2)^{1/2}$. The metric functions $\mu(r)=1-2m(r)/r$ of the dyon solutions develops a minimum which decreases monotonically along the fundamental solutions,

$$\mu(r)=1-2m(\infty)/r+\alpha^2/v^2r^2 \tag{2.26}$$

where $m(_\infty)=m_\infty=\alpha/v=\sqrt{G}=r_H$.

For $\alpha\to\alpha_c$ this function has a zero for

$$1-2\alpha_c/vr+\alpha_c^2/v^2r^2=0$$

which gives a double zero of $\mu(r)$ at

$$r=\alpha_c/v=r_c. \tag{2.27}$$

Thus the limiting functions consists of an inner part $r\le r_c$ or $r\le\alpha_c/v$ and an outer part $r\ge\alpha_c$ or $r\ge r_c/v$ for which we have $\mu(r)$ of the extremal R-N black hole. Then we have

$$J(r)=J_\infty-e/r \quad \text{and} \quad H(r)=v(1+e^2)^{1/2} \tag{2.28}$$

showing that $J(r)$ reaches the limiting function at $\alpha\to\alpha_c$. The limiting function is identically zero for $r\le r_c$ and coincides with R-N functions for $r\ge r_c$ [23]. The limiting behaviour of other functions of dyonic solutions is similar to those of monopole solutions [10, 11].

In addition to fundamental gravitating dyonic solutions, leading to its flat limit for $\alpha \to 0$, there exists a discrete family of radically excited dyon solutions having no counterpart in flat space. There are n nodes of the gauge field function $K(r)$ of the nth excited dyon solution. For these solutions the variations of α is obtained by varying v while keeping G fixed. In the limit $\alpha \to 0$ the excited solutions shrink to zero size with diverging mass. For $\alpha > 0$ and any $n \geq 0$, there exists at least one global quasi regular solution with n zeroes of $K(r)$. For the same value of α there exists at least one oscillatory solution with infinitely many zeroes of $K(r)$ and $0 \leq r \leq 1$.

(C) Black-hole Solutions

Another new feature without any analogue in the flat space is existence of non-Abelian dyonic black holes which exist in a certain bounded domain in (α, r_H) plane, where r_H is the radius of black hole horizon. The precise form of this domain depends on β. For the existence of a regular even horizon at r_H, we require

$$m(r_H) = r_H/2, \quad \text{and}$$
$$A(r_H) < \infty$$

with the matter function satisfying

$$\begin{aligned}
&\lim_{r\to r_H} (\mu' K') = \lim_{r\to r_H} [K(K^2-1)/r^2 + KH^2]/(v^2|q|^2),\\
&\lim_{r\to r_H} J(r) = 0 \quad \text{and}\\
&\lim_{r\to r_H} [r^2\mu' H'] = \lim_{r\to r_H} [2HK^2 + \beta^2 H(H^2-|q|^2v^2)]/(|q|^2v^2).
\end{aligned} \tag{2.29}$$

These equations lead to black holes within the dyon in the manner similar to the existence of black holes within the monopoles [10, 11]. For a given α, the black hole solutions emerge from the globally regular solution in the limit $r_H \to 0$ and persist upto a critical maximal value of the horizon radius as shown by Brihaye et al. [23]. For small values of α, the black hole solutions merge into non-extremal R-N solutions at a critical value of the horizon radius at $\alpha = \alpha_c$. For larger α the black hole solutions bifurcate with an extremal R-N solution.

In view of quantization condition (2.14a) for dyons, the mass

$$m(r) = m_\infty - G/2r \tag{2.30}$$

can not be lowered continuously by lowering the electric charge alone and hence the gravitational fundamental dyon solutions should also the stable (like monopole solutions) in contrast to the result of Brihaye et al. [23]. This stability of dyonic solutions is consistent with the quantization condition (2.14a).

3 Gravitating Dyons in SU(5) Theory

In the SU(2) case with the action given by (2.5), the gravitating solutions representing static dyon–antidyon pairs are at most axially symmetric [32] consistent with the flat limit. In SU(5) theory, $V(\phi)$ of action given by (2.5) is SU(5) potential which may be written as [24]

$$\begin{aligned}
V(\phi, H) = {}& -\lambda_1 t_r(\phi^2) + \lambda_2(tr\phi^2)^2 + \lambda_3 t_r(\phi^4)\\
&+ \lambda_4(H^\dagger H - \omega^2)^2 + \lambda_5 H^\dagger(\phi + 3v/2\sqrt{15})^2 H - V_{\min}
\end{aligned} \tag{3.1}$$

when ϕ is 24-representation of SU(5) and H is the 5-representation. In this case (2.2) may be written as

$$\vec{V}_\mu = V_\mu^a \lambda_a, \quad a = 1, 2, \ldots, 24,$$

where the group generators λ^α satisfy the relations

$$t_r \lambda^a \lambda^b = 1/2\delta^{ab}, \quad \lambda^{\dagger a} = \lambda^a.$$

The group SU(5) is spontaneously broken down to SU(3) XU(1)$_{\text{e.m.}}$ by the following vacuum expectation values of ϕ and H;

$$\langle\phi\rangle = v\,\mathrm{diag}(1/\sqrt{15}, 1/\sqrt{15}, 1/\sqrt{15}, -3/2\sqrt{15}, -3/2\sqrt{15})$$

and

$$\langle H\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \omega \end{pmatrix}. \tag{3.2}$$

In (3.1) the term

$$V_{\min} = -(\lambda_2 + 5\lambda_3)v^4/20$$

has been fixed due to the finiteness of the energy. Under the boundary conditions of finite energy and constant Higgs field

$$\phi(0, 0, \infty) = i\phi_0 \tag{3.3}$$

at infinity (in x_3-direction), we have

$$\begin{gathered} \phi_0 = \mathrm{diag}(k_1, k_2, k_3, k_4, k_5) \\ \text{where} \sum_{i=1}^{5} ki = 0 \end{gathered} \tag{3.4}$$

and the difference

$$k_j - k_{j+1} = m_j$$

determines the mass of dyon of type-j.

Then the action for SU(5) gravitating dyons is

$$\begin{aligned} S = \int [R/16\pi G - (1/2)t_r(G\mu\upsilon G^{\mu\upsilon}) - t_r(D_\mu\phi D^\mu\phi) \\ + g^{\mu\upsilon}(D_\mu H)^\dagger(D_\upsilon H) - V(\phi, H)]\sqrt{-g'}d^4x \end{aligned} \tag{3.5}$$

with $G_{\mu\upsilon}$ given by (2.3);

$$\begin{aligned} D_\mu H &= \partial_\mu H + |q|V_\mu H, \\ D_\mu\phi &= \partial_\mu\phi + |q|[V_\mu, \phi]. \end{aligned} \tag{3.6}$$

This action yields the Einstein equation

$$R\mu\nu - \frac{1}{2}g\mu\nu R = 8\pi G T\mu\nu \tag{3.7}$$

where

$$\begin{aligned} T\mu\nu &= g_{\mu\nu}L - 2\partial L/\partial g_{\mu\nu} \\ &= 2t_r(D_\mu\phi D_\upsilon\phi) - g_{\mu\upsilon}t_r(D_\alpha\phi D^\alpha\phi) \\ &\quad + 2t_r(g^{\alpha\beta}G_{\mu\alpha}G_{\upsilon\beta}) - \frac{1}{2}g_{\mu\upsilon}t_r(G_{\alpha\beta}G^{\alpha\beta}) \\ &\quad + 2g_{\mu\rho}(D^\rho H)^\dagger(D_\upsilon H) - g_{\mu\upsilon}g^{\rho\alpha}(D_\rho H)^\dagger(D_\sigma H) \\ &\quad - g_{\mu\upsilon}V(\phi, H). \end{aligned} \tag{3.8}$$

For spherically symmetric static dyon, we adopt the space–time metric

$$ds^2 = e^A dt^2 - e^B dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.9}$$

where A and B are functions of r only. The function e^A has a geometrical significance as the length of the time translation Killing vector and the function e^B is gauge dependent i.e. it depends on the choice of the radial coordinate r. Einstein equation (3.7) imposes boundary conditions for $A(r)$ and $B(r)$.

For SU(5) case (2.9) may be written as [32];

$$\begin{aligned} H(\vec{r}) &= 1/|q|\mathrm{Col}(0,0,0,0,h), \\ V^{ia}(\vec{r}) &= \delta^{ai}_{kl}T^k x^l[K(r)-1]/r^2|q|, \\ \phi(\vec{r}) &= (1/|q|)\mathrm{diag}[\phi_1(r), \phi_1(r), \phi_2(r) + (1/2)\phi_3(r)\hat{r}.\tau - 2\{\phi_1(r)+\phi_2(r)\}] \quad \text{and} \\ \vec{V}_0(r) &= (1/|q|)\mathrm{diag}[J_1(r), J_1(r)J_2(r) + (1/2)J_3(r)\hat{r}.\tau - 2\{J_1(r)+J_2(r)\}] \end{aligned} \tag{3.10}$$

where T^k, the generators of SU(2) embedding in to SU(5) with $5—2+1+1+1$, are given by

$$T = (1/2)\mathrm{diag}(0,0,\tau,0),$$

with τ as set of Pauli matrices,

$$\hat{r} = (\vec{r})/|\vec{r}|,$$

and

$$\delta^{ai}_{kl} = \delta^a_k\delta^i_l - \delta^a_l\delta^i_k$$

is the generalized invariant delta symbol.

This form of the field configuration is time independent and spherically symmetric under $\vec{J}\times\vec{T}$ where $\vec{J}$, the gauge invariant and rotationally symmetric momentum operators for dyons is given by [33];

$$J = \vec{r}\times(\vec{p} - \mu_{ij}V^T) + \mu_{ij}\vec{r}/r,$$

with $\vec{V}^T$ as transverse generalized vector potential for dyons.

Einstein equation (3.7) then gives

$$
\begin{aligned}
e^{-B(r)}[rB' - 1]/r^2 + 1/r^2 &= 8\pi G T_0^0, \\
-e^{-B(r)}[rA' + 1]/r^2 + 1/r^2 &= 8\pi G T_r^r, \\
-e^{-B(r)}[A'' + A'^2/2 - A'B'/2 + (A' - B')/r] &= 16\pi G T_\theta^\theta = -16\pi G T_\phi^\phi.
\end{aligned}
\tag{3.11}
$$

From action, given by (3.5), we also get following field equations;

$$
\begin{aligned}
&(\sqrt{-g})^{-1} D^v(\sqrt{-g} G_{\mu v}) = |q|[\phi, D_\mu \phi] + |q|\lambda^a[H^\dagger \lambda^a D_\mu H - (D_\mu H)^\dagger \lambda^a H], \\
&(\sqrt{-g})^{-1} D_\mu(\sqrt{-g} D^\mu \phi) = -\partial V/\partial \phi, \\
&(\sqrt{-g})^{-1} D_\mu[(\sqrt{-g}) D^\mu H] = -(1/2)\partial V/\partial H^\dagger.
\end{aligned}
\tag{3.12}
$$

Substituting relations (3.10) in these equations, we get the following radial equations;

$$
\begin{aligned}
&(rJ_i)'' - (1/2)(A+B)'J_i' + (2/5)re^B \sum_{J=1}^{3} J_i h^2 - (2/5)e^B[rJ_i h^2 + 5J_i K^2]\delta_i^3 = 0, \\
&K'' + (1/2)(A-B)'K' - (e^B/r^2)K(K^2-1) - 4e^B K\phi_3^2 + e^{-(A-B)}KJ_3^2 = 0, \\
&(r\phi_I)'' + (1/2)r(A-B)'\phi_i' - 2e^B K^2/r^2 \phi_i \delta_i^3 = (1/2)|q|^2 re^B \partial V/\partial \phi_i, \\
&(rh)'' + (1/2)r(A-B)'h' = -4re^{(B-A)}(J_1+J_2)^2 h + |q|^2 re^B \partial V/\partial h
\end{aligned}
\tag{3.13}
$$

where dash denotes derivative with respect to r.

Substituting relations (3.10), (3.11) yield following relations

$$
\begin{aligned}
T_0^0 = T_t^t &= e^{-(A+B)}T_1/|q|^2 + e^{-A}[4T_2 + J_3^2K^2/r^2]/|q|^2 \\
&\quad + e^{-B}[T_3 + K'^2/r^2]/|q|^2 + 4\phi_3^2K^2/(|q|^2r^2) + (K^2-1)^2/(2r^4|q|^2) \\
&\quad + h'^2/(|q|^2r^2) + V(\phi, H), \\
T_0^0 - T_r^r &= 2e^{-A}[4T_2 + J_3^2K^2/r^2]/|q|^2 + 2e^{-B}[T_3 + K'^2/r^2]/|q|^2 + 2h'^2/(|q|^2r^2), \\
T_\theta^\theta = -T_\phi^\phi &= e^{-(A+B)}T_1/|q|^2 - 4e^{-A}T_2/|q|^2 + e^{-B}T_3/|q|^2 \\
&\quad + (K^2-1)^2/(2|q|^2r^4) + h'^2/r^2 + V(\theta, H),
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
T_1 &= 6J_1'^2 + 6J_2'^2 + 8J_1'J_2' + J_3'^2/2, \\
T_2 &= (J_1 + J_2)^2 h^2, \\
T_3 &= 6\phi_1'^2 + 6\phi_2'^2 + 8\phi_1'\phi_2' + 2\phi_3'^2,
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
V(\phi, H) = &-2\lambda_1(3\phi_1^2 + 3\phi_2^2 + 4\phi_1\phi_2 + \phi_3^2)/|q|^2 \\
&+ 4\lambda_2[3\phi_1^2 + 3\phi_2^2 + 4\phi_1\phi_2 + \phi_3^2]^2/|q|^2 \\
&+ 2\lambda_3[9\phi_1^4 + 9\phi_2^4 + 32\phi_1^3\phi_2 + 48\phi_1^2\phi_2^2 + 32\phi_1\phi_2^3 \\
&+ 6\phi_2^2\phi_3^2 + \phi_3^4]^2/|q|^4 - V_{\min}.
\end{aligned}
\tag{3.16}
$$

The properties of a given solution can be read off by the values of Higgs field of (3.10) at $\vec{x} = (0, 0, \infty)$, which corresponds to the direction $\theta = 0$;

$$\phi_0 = (1/|q|)\text{diag}[\phi_1, \phi_2, \phi_2 + \phi_3, \phi_2 - \phi_3, -2(\phi_1 + \phi_2)]. \tag{3.17}$$

Under the boundary condition that this Higgs field at spatial infinity approaches the Higgs vacuum given by equation (3.2), we have

$$h/|q| = \omega \quad \text{or} \quad h = \omega|q|, \tag{3.18}$$

and also

$$\phi_1/|q| = \upsilon/\sqrt{15}, \qquad (\phi_2 + \phi_3)/|q| = \upsilon/\sqrt{15}, \qquad (\phi_2 - \phi_3)/|q| = -3\upsilon/2\sqrt{15},$$
$$\text{and} \quad (\phi_1 + \phi_2)/|q| = 3\upsilon/4\sqrt{15}.$$

These relations give

$$\phi_1 = \upsilon|q|/\sqrt{15}, \qquad \phi_2 = -\upsilon|q|/4\sqrt{15}, \qquad \phi_3 = 5\upsilon|q|/4\sqrt{15}. \tag{3.19}$$

Then we have

$$V(\phi, H) \to 0.$$

For usual R-N metric there exists the relation

$$T_t^t = T_r^r.$$

Then (3.12) and (3.13) yield following relations;

$$T_2 = T_3 = 0, \qquad h' = 0, \qquad J_3 K = 0 \quad \text{and} \quad K' = 0$$

which give

$$\begin{aligned} &(J_1 + J_2)h = 0, \quad \phi_i' = 0 \quad (i = 1, 2, 3), \\ &h' = 0, \qquad K' = 0 \quad \text{and} \\ &J_3 K = 0. \end{aligned} \tag{3.20}$$

But $h = \omega|q| \neq 0$ and hence the first one of these relations gives

$$(J_1 + J_2) = 0 \quad \text{and} \quad J_i \neq 0. \tag{3.21}$$

$K = 0$ and ϕ_i given by (3.19) for dyonic solutions. Further-more, from (3.11) we also have

$$T_t^t = T_r^r = e^{-B(r)}[(A + B)'/r]$$

and hence R-N metric requires

$$A + B = C \quad (\text{const}).$$

Setting $C = 0$ to make the metric asymptotically flat, we have

$$A = -B. \tag{3.22}$$

In view of conditions (3.20) and (3.21), we may set

$$J_1 = a_1/r + c_1, \qquad J_2 = a_2/r + c_2, \qquad J_3 = a_3/r + c_3 \tag{3.23}$$

when a_1, a_2, c_1 and c_2 are constants with $a_1 = -a_2$ and $c_1 = -c_2$.

Substituting these relations into (3.14) and (3.15), we get

$$\begin{aligned} T_t^t &= T_r^r = (2a_1^2 + 2a_2^2)/|q|^2 r^4 + (a_3^2 + 1)/2|q|^2 r^4, \quad \text{or} \\ T_t^t &= T_r^r = (4a_1^2 + 4a_2^2 + a_3^2 + 1)/2r^4|q|^2. \end{aligned} \tag{3.24}$$

Substituting it in to relations (3.11), we have

$$e^{-B(r)}[rB' - 1] + 1 = 4\pi G[4a_1^2 + 4a_2^2 + a_3^2 + 1]/|q|^2 r^2$$

or

$$(d/dr)(-e^{-B(r)}r + r) = 4\pi G[4a_1^2 + 4a_2^2 + a_3^2 + 1]/|q|^2 r^2$$

or

$$e^{-B(1)} = 4\pi G[4a_1^2 + 4a_2^2 + a_3^2 + 1]/|q|^2 r^2 - D/r + 1,$$

where D is the constant of integration. Setting this constant as $2GM_D$ where M_D is the mass of dyon, we have

$$e^{-B(r)} = e^{A(r)} = 1 - 2M_D G/r + 4\pi G[4a_1^2 + 4a_2^2 + a_3^2 + 1]/|q|^2 r^2. \tag{3.25}$$

With these values the metric given by (3.9) is asymptotically flat. With this asymptotically flat metric, the electric and magnetic filed can be integrated over a sphere at infinity and the corresponding charges can be calculated . Let us define the electromagnetic field strength as

$$F\mu\upsilon(r) = \sqrt{(3/2)}t_r[G\mu\upsilon(\vec{r})Q(\vec{r})] \tag{3.26}$$

where

$$Q(\vec{r}) \underset{r\to\infty}{\to} \text{diag}\big[-1/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6}, -2\vec{r}.\tau/\sqrt{6}, 0\big].$$

Then the magnetic field $B(\vec{r})$ and the electric field $E(\vec{r})$ may be written as

$$\begin{aligned} B_i(\vec{r}) &= 1/2\sqrt{(3/2)}t_r[\varepsilon_{ijk}G_{jk}(\vec{r})Q(\vec{r})], \\ E_i(\vec{r}) &= F_{oi}(\vec{r}) = \sqrt{(3/2)}t_r[G_{oi}(\vec{r})Q(\vec{r})]. \end{aligned} \tag{3.27}$$

Substituting relations (3.10), (3.21), (3.23) and (2.3) into these equations, we get

$$\vec{B}(\vec{r}) \underset{r\to\infty}{\to} \sqrt{(3/2)}\hat{r}/|q|r^2 \quad \text{and} \quad \vec{E}(\vec{r}) \underset{r\to\infty}{\to} \sqrt{(2/3)}(-a_1 + a_2 - a_3)/|q|\hat{r}/r^2$$

and hence the magnetic charge g and the electric charge e of dyon may be written as

$$\begin{aligned} &g = 4\pi\sqrt{(2/3)}/|q| \quad \text{and} \\ &e = 4\pi\sqrt{(2/3)}(-a_1 + a_2 - a_3)/|q| = 1/2e_1 + 1/2e_2 + e_3 \\ &\quad \text{where } e_1 = -8\pi\sqrt{(2/3)}a_1/|q|, \ e_2 = 8\pi\sqrt{(2/3)}a_2/|q|, \\ &\quad \text{and } e_3 = -4\pi\sqrt{(2/3)}a_3/|q|. \end{aligned}$$

Substituting these relations into (3.25), we get

$$e^{-B} = e^{A} = 1 - 2M_D G/r + (3G/8\pi r^2)(g^2 + e_1^2 + e_2^2 + e_3^2) \tag{3.28}$$

and hence the asymptotically flat metric of dyon may be written as follows from (3.9);

$$\begin{aligned} ds^2 = {} & [1 - 2GM_D/r + (3G/8\pi r^2)(g^2 + e_1^2 + e_2^2 + e_3^2)]dt^2 \\ & - [1 - 2GM/r + (3G/8\pi r^2)(g^2 + e_1^2 + e_2^2 + e_3^2)]^{-1} dr^2 \\ & - r^2(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \tag{3.29}$$

The corresponding metric for monopole may be written by setting $e_1 = e_2 = e_3 = 0$. Thus the space–time, for monopoles as well as dyons, is asymptotically flat.

The field equations (3.13) have to be complemented with suitable boundary conditions, different from those given by (3.17), (3.18) and (3.19), for getting the solutions with regular origin or black holes. In both these cases the corresponding boundary points are singular points of the field equations. For $r = 0$, it is obvious from (3.13) but it is not so obvious in the case of black holes.

When the solutions with flat metric are coupled to gravity (i.e. $\alpha \neq 0$), they deform gravitating configuration and their presence progressively deforms space–time.

In the way similar to that discussed by Brihaye et al. [21] for SU(5) gravitating monopoles the SU(5) gravitating dyons also bifurcates into an extremal Reisser-Nordstrom black hole. This configuration corresponds to the solution of the Abelian-Einstein-Maxwell equations and can be embedded into the non-Abelian one with the mass given by (2.30). In view of chirality quantization condition (2.14a), this mass can not be lowered by lowering the electric charge and hence such solutions in SU(5) theory also should be stable.

There may be many minimal symmetry breaking patterns producing SU(5) solution with non-Abelian stability group. These solutions can be deformed by gravity forming branches of solutions labeled by the gravitational couplings parameter α. For each branch of solutions, a second branch exists on the interval $\alpha_c < \alpha < \alpha_{\max}$ in the similar manner as discussed in earlier section for SU(2) theory.

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